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Nonlinear Eigenvalue Problems and Critical Points of Functions

JAMES M. GREENBERG

Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

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1. INTRODUCTION

We here study the nonlinear eigenvalue problem

$$A\mathbf{x} + \mathbf{F}(\mathbf{x}) = \lambda\mathbf{x} \tag{1.1}$$

where $A : R^n \to R^n$ is self adjoint and linear and $\mathbf{F}(\cdot)$ is the gradient of a potential Ψ; i.e.

$$\mathbf{F}(\mathbf{x}) = \nabla_{\mathbf{x}}\Psi(\mathbf{x}). \tag{1.2}$$

It is well known that the nontrivial solutions of (1.1) of fixed amplitude $r = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ are the critical points of $\phi(\mathbf{x}) = \frac{1}{2} A\mathbf{x} \cdot \mathbf{x} + \Psi(\mathbf{x})$ on $\mathbf{x} \cdot \mathbf{x} = r^2$. Moreover, if $\mathbf{y}$ is such a critical point, then the eigenvalue $\lambda(\mathbf{y})$ is given by

$$\lambda(\mathbf{y}) = \frac{(A\mathbf{y} + \nabla_{\mathbf{x}}\Psi(\mathbf{y})) \cdot \mathbf{y}}{r^2}. \tag{1.3}$$

It is no loss in generality to assume that

$$\tfrac{1}{2} A\mathbf{x} \cdot \mathbf{x} = \sum_{i=1}^{n} \lambda_i x_i^2; \tag{1.4}$$

in this case ϕ takes the form

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i x_i^2 + \Psi(\mathbf{x}). \tag{1.5}$$

Our interest is in showing that if appropriate conditions are met, then the nontrivial solutions of (1.1) (eigenvectors of $A + F$) may be parameterized smoothly by r.

We also obtain results about the maximal extension of a given branch of eigenvectors.

2. Statement of Results

Let ϕ be given by (1.4) and assume that the numbers $\{\lambda_j, j = 1, 2,..., n\}$ are distinct and indexed in decreasing order ($\lambda_j > \lambda_{j+1}$). In addition assume that the map $\mathbf{x} \to \Psi(\mathbf{x}) : R^n \to R$ is smooth (C^3 will suffice) and satisfies

$$\left(\sum_{|\alpha|=j} | D^\alpha \Psi(\mathbf{x}) |^2\right)^{1/2} \leqslant K \| \mathbf{x} \|^{3-j}, \qquad j = 0, 1, 2. \tag{2.1}$$

In (2.1) D^α stands for any derivative of order j and $\| \cdot \|$ for the Euclidean norm.

The assumption that the λ_j's are distinct implies that the vectors $\pm r\mathbf{e}_j$ with

$$\mathbf{e}_j = (\overbrace{0,..., 0}^{j-1}, \overset{j}{1}, \overbrace{0,..., 0}^{n-j}), \qquad j = 1, 2,..., n,$$

are the unique critical points of $\phi_0 \equiv \sum_{i=1}^n \lambda_i x_i^2$ on $\mathbf{x} \cdot \mathbf{x} = r^2$.

For each $0 < \epsilon < 1$ we let

$$\eta_j^{+(-)}(1, \epsilon) = \left\{\mathbf{v} \,\middle|\, v_j = \binom{+}{-} \sqrt{1 - \sum_{\substack{k=1\\k\neq j}}^n v_k^2}, \sum_{\substack{k=1\\k\neq j}}^n v_k^2 \leqslant \epsilon^2\right\}.[1] \tag{2.2}$$

For our purposes we will want two numbers $0 < \epsilon_1 < \epsilon_2 < 1$ such that the neighborhoods $\{\eta_j^{+(-)}(1, \epsilon_1), j = 1, 2,..., n\}$ are disjoint on the unit sphere while the neighborhoods $\{\eta_j^{+(-)}(1, \epsilon_2), j = 1, 2,..., n\}$ cover the unit sphere.

Theorem 1 (Local Existence and Uniqueness Theorem). *There is an $r_0 > 0$ such that for any $r \in (0, r_0]$ the function ϕ has exactly $2n$ critical points on the sphere $\mathbf{x} \cdot \mathbf{x} = r^2$. These points may be labeled in pairs*

$$(\mathbf{x}_j^+(r), \mathbf{x}_j^-(r)) \equiv r(\mathbf{v}_j^+(r), \mathbf{v}_j^-(r))$$

according to the scheme

$$\mathbf{v}_j^+(r) \in \eta_j^+(1, \epsilon_1) \qquad \textit{and} \qquad \mathbf{v}_j^-(r) \in \eta_j^-(1, \epsilon_1). \tag{2.3}$$

The functions $r \to \mathbf{v}_j^+(r)$ (respectively $r \to \mathbf{v}_j^-(r)$) are $C^1(0 < r \leqslant r_0)$ and satisfy

$$\lim_{r\to 0} \mathbf{v}_j^+(r) = \mathbf{e}_j \qquad \textit{and} \qquad \lim_{r\to 0} \mathbf{v}_j^-(r) = -\mathbf{e}_j, \qquad j = 1, 2,..., n. \tag{2.4}$$

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[1] or equivalently $\eta_j^{+(-)}(1, \epsilon) \equiv \{\mathbf{v} \mid \| \mathbf{v} \|^2 = 1, \| \mathbf{v} \mp_{(+)} \mathbf{e}_j \|^2 \leqslant \epsilon^2\}$

In order to state the global existence theorem it is necessary to introduce some additional notation. For each $\mathbf{v} \ni \|\mathbf{v}\|^2 = 1$ we let $V(\mathbf{v})$ be the $n - 1$ dimensional vector space

$$V(\mathbf{v}) \equiv \{\mathbf{u} \in R^n \mid \mathbf{u} \cdot \mathbf{v} = 0\}. \tag{2.5}$$

For each $r > 0$ and $\mathbf{v} \ni \|\mathbf{v}\|^2 = 1$ we define the symmetric bilinear form $B(r\mathbf{v}; \cdot, \cdot) : V(\mathbf{v}) \times V(\mathbf{v}) \to R$ by

$$B(r\mathbf{v}; \mathbf{u}, \mathbf{w}) \equiv \frac{1}{r^2} \frac{\partial^2}{\partial s\, \partial t} \phi(r(\mathbf{v} + s\mathbf{u} + t\mathbf{w})) \Big|_{s=t=0} - \frac{1}{r} (\nabla_{\mathbf{x}} \phi(r\mathbf{v}) \cdot \mathbf{v}) (\mathbf{u} \cdot \mathbf{w}). \tag{2.6}$$

$\mathscr{B}(r\mathbf{v}) : V(\mathbf{v}) \to V(\mathbf{v})$ is the linear operator associated with the bilinear form $B(r\mathbf{v}; \cdot, \cdot)$.[2]

THEOREM 2 (Global Existence of a Given Branch of Critical Points).[3] *For each j it is possible to extend the function $r \to \mathbf{v}_j^+(r)$ from $[0, r_0]$ to some maximal interval $[0, R_j^+)$ in such a way that the function $\mathbf{x}_j^+(r) \equiv r\mathbf{v}_j^+(r)$ is a critical point of ϕ (or $\mathbf{x} \cdot \mathbf{x} = r^2$). The function $\mathbf{v}_j^+(\cdot)$ is extended as the unique solution of the initial value problem:*

$$\begin{aligned} \mathscr{B}(r\mathbf{v})\, \dot{\mathbf{v}}(r) &\equiv \mathbf{F}(r, \mathbf{v}), \qquad r > r_0 \\ \mathbf{v}(r_0) &\equiv \mathbf{v}_j^+(r_0). \end{aligned} \tag{2.7}$$

The initial data $r_0 \mathbf{v}_j^+(r_0)$ is the unique critical point of ϕ on $\mathbf{x} \cdot \mathbf{x} = r_0^2$ such that $\mathbf{v}_j^+(r_0) \in \eta_j^+(1, \epsilon)$, and $\mathbf{F}(r, \mathbf{v}) \in V(\mathbf{v})$ is defined by

$$F(r, \mathbf{v}) \equiv \frac{\partial}{\partial r} \left\{ \frac{1}{r} [\nabla_{\mathbf{x}} \Psi(r\mathbf{v}) - (\nabla_{\mathbf{x}} \Psi(r\mathbf{v}) \cdot \mathbf{v})\, \mathbf{v}] \right\}.$$

The number R_j^+ is characterized as the first $r > 0$ such that the quadratic form $B(r\mathbf{v}_j^+(r); \mathbf{u}, \mathbf{u})$ has zero as a critical value on the unit sphere $V(\mathbf{v}_j^+(r))$. For all $r < R_j^+$ the quadratic has $j - 1$ positive and $n - j$ negative critical values on the unit sphere $V(\mathbf{v}_j^+(r))$.

The following example shows that Theorems 1 and 2 are the best that may be expected.

Let

$$\phi_0 = x^2 + 2y^2,$$
$$\Psi = \tfrac{2}{3} x^3 - (x^2 + y^2)(x^2 + 2y^2),$$

and

$$\phi = \phi_0 + \Psi.$$

[2] For any $\mathbf{u} \in V(\mathbf{v})$ $B(r\mathbf{v})\mathbf{u} = [\nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \phi(r\mathbf{v})]\mathbf{u} - (\mathbf{u} \cdot [\nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \phi(r\mathbf{v})]\, \mathbf{v})\mathbf{v} - 1/r\, (\nabla_{\mathbf{x}} \phi(r\mathbf{v}) \cdot \mathbf{v})\mathbf{u}$.

[3] It has been called to the authors' attention that a similiar unpublished result has been obtained by H. T. Sedinger.

If we introduce the polar coordinates

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta,$$

then

$$\eta(r, \theta) \equiv \frac{\phi}{r^2}(r\cos\theta, r\sin\theta)$$

takes the form

$$\eta(r, \theta) \equiv (1 - r^2)(\cos^2\theta + 2\sin^2\theta) + \tfrac{2}{3} r\cos^3\theta.$$

A simple computation shows that the critical points of η on $x^2 + y^2 = r^2$ are those numbers $\theta \in [0, 2\pi)$ which satisfy

$$\frac{\partial\eta}{\partial\theta}(r, \theta) = \sin 2\theta(1 - r^2 - r\cos\theta) = 0.$$

It is clear that for all $r > 0$ the numbers $\theta = 0, \pi/2, \pi$, and $3\pi/2$ are critical points of η. For

$$\frac{\sqrt{5}-1}{2} < r < \frac{\sqrt{5}+1}{2}$$

there are two additional critical points $\theta_1 \in (0, \pi)$ and $\theta_2 \in (\pi, 2\pi)$ which satisfy

$$\cos\theta_i = \frac{1-r^2}{r}, \qquad \frac{\sqrt{5}-1}{2} < r < \frac{\sqrt{5}+1}{2}, \qquad i = 1, 2.$$

Finally, for $r > (\sqrt{5}+1)/2$ the numbers $\theta = 0$, $(\pi/2)$, π, and $3\pi/2$ are again the only critical points.

We now analyze these critical points in more detail. For $0 < r < (\sqrt{5}-1)/2$ the points $\theta = 0$ and π correspond to relative minima of η while $\theta = \pi/2$ and $3\pi/2$ correspond to relative maxima. At $r = (\sqrt{5}-1)/2$ (the point where $\partial^2\eta/\partial\theta^2|_{\theta=0} = 0$) the point $\theta = 0$ becomes an inflection point, and for all $r > (\sqrt{5}+1)/2$, $G \equiv 0$ corresponds to a maximum of η.

For $(\sqrt{5}-1)/2 < r < 1$ the points $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (3\pi/2, 2\pi)$ correspond to relative minima of η and the character of the points $\theta = \pi/2$, π, and $3\pi/2$ is as before. At $r = 1$ (the point where $\partial^2\eta/\partial\theta^2|_{\theta=\pi/2} = \partial^2\eta/\partial\theta^2|_{\theta=3\pi/2} = 0$), the point θ_1 coalesces with $\pi/2$ and θ_2 with $3\pi/2$. For $1 < r < (\sqrt{5}+1)/2$, the points $\theta = \pi/2$ and $3\pi/2$ are relative minima and the points θ_1 and θ_2 are relative maxima. $\theta = \pi$ is still a relative minimum. At $r = (\sqrt{5}+1)/2$ (the place where $\partial^2\eta/\partial\theta^2|_{\theta=\pi} = 0$) $\theta_1 = \theta_2 = \pi$ is an inflection point of η. For $r > (\sqrt{5}+1)/2$ the critical points $\theta = 0$ and π

correspond to relative maxima of η while $\theta = \pi/2$ and $3\pi/2$ correspond to relative minima.

3. Proofs

To establish Theorem 1 we look at ϕ in neighborhoods of the critical points of $\phi_0 \equiv \sum_{i=1}^{n} \lambda_i x_i^2$ on $\mathbf{x} \cdot \mathbf{x} = r^2$. Clearly, it suffices to look at ϕ in a neighborhoods of $r\mathbf{e}_1$.

We now let $0 < \epsilon_1 < \epsilon_2 < 1$ be two numbers such that the neighborhoods $\eta_j^{+(-)}(1, \epsilon_1)$, $j = 1, 2, \ldots, n$ are disjoint on $\mathbf{v} \cdot \mathbf{v} = 1$ while the neighborhoods $\eta_j^{+(-)}(1, \epsilon_2)$, $j = 1, 2, \ldots, n$ cover $\mathbf{v} \cdot \mathbf{v} = 1$. We set $\mathbf{x} = r\mathbf{v}$, $\mathbf{v} \equiv (v_1, v_2, \ldots, v_n)$ and introduce local coordinates:

$$v_1 = \sqrt{1 - \sum_{k=2}^{n} v_k^2}$$

where

$$\sum_{k=2}^{n} v_k^2 \leqslant \epsilon_2^2. \tag{3.1}$$

The function ϕ becomes $r^2\hat{\phi}(r, \mathbf{v})$ where

$$\hat{\phi}(v_2, v_3, \ldots, v_n; r) \equiv \left[\lambda_1 + \sum_{k=2}^{n} \mu_k{}^1 v_k{}^2\right] + \frac{1}{r^2} \Psi\left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n\right), \tag{3.2}$$

and

$$\mu_k{}^1 = \lambda_k - \lambda_1 .$$

To obtain Theorem 1 it suffices to show that for r sufficiently small ($\leqslant$ some r_0)

(A) There exists a unique $n - 1$ tupple $(v_2{}^+, v_3{}^+, \ldots, v_n{}^+)$ with $\sum_{k=2}^{n} v_k^2 \leqslant \epsilon_1^2$ such that

$$\frac{\partial \hat{\phi}}{\partial v_i}(v_2, v_3, \ldots, v_n; r) = \left\{2\mu_i{}^1 - \frac{\Psi_{x_1}\left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n\right)}{r\sqrt{1 - \sum_{k=2}^{n} v_k^2}}\right\} v_i + \frac{1}{r} \Psi_{x_i}\left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n\right) \equiv 0, \qquad i = 2, 3, \ldots, n. \tag{3.3}$$

(B) The $(n-1)$ tupple $(v_2^+, v_3^+, \ldots, v_n^+)$ satisfying (3.3) is the only solution in the larger sphere $\sum_{k=1}^n v_k^2 \leqslant \epsilon_2^2$; and

(C) the map $r \to (v_2^+(r), v_3^+(r), \ldots, v_n^+(r))$ is C^1 $(0 < r \leqslant r_0)$ and satisfies

$$\lim_{r\to 0} (v_2^+(r), v_3^+(r), \ldots, v_n^+(r)) = (0, 0, \ldots, 0). \tag{3.4}$$

The growth condition on Ψ implies that

$$\left| \frac{1}{r} \frac{\Psi_{x_1}\left(r\sqrt{1 - \sum_{k=2}^n v_k^2}, rv_2, \ldots, rv_n\right)}{\sqrt{1 - \sum_{k=2}^n v_k^2}} \right| \leqslant \frac{Kr}{\sqrt{1-\epsilon_2^2}} \tag{3.5}$$

and

$$\left(\sum_{i=2}^n \left| \frac{\Psi_{x_i}\left(r\sqrt{1 - \sum_{k=2}^n v_k^2}, rv_2, \ldots, rv_n\right)}{r} \right|^2 \right)^{1/2} \leqslant Kr \tag{3.6}$$

for all $(v_2, v_3, \ldots, v_n)$ satisfying

$$\sum_{k=2}^n v_k^2 \leqslant \epsilon_2^2. \tag{3.7}$$

Equation (3.5) implies that if

$$r \leqslant r_1 \equiv \frac{\min_i |\mu_i^1| \sqrt{1-\epsilon_2^2}}{K} = \frac{|\mu_2^1| \sqrt{1-\epsilon_2^2}}{K}, \tag{3.8}$$

and if (3.7) holds, then the operator

$$P \equiv \operatorname{diag}(p_1, p_2, \ldots, p_n)$$

with

$$p_i = 2\mu_i^1 - \frac{\Psi_{x_1}\left(r\sqrt{1 - \sum_{k=2}^n v_k^2}, rv_2, \ldots, rv_n\right)}{r}$$

is invertible and

$$\| P^{-1} \| \leqslant \frac{1}{|\mu_2^1|}. \tag{3.9}$$

It now follows that solving (3.3) is equivalent to solving

$$v_i = T_i(v_2, v_3, \dots, v_n; r), \qquad i = 2, 3, \dots, n \tag{3.10}$$

where

$$T_i(v_2, v_3, \dots, v_n; r) \equiv \frac{r^{-1}}{2\mu_i^1 - \dfrac{\Psi_{x_1}}{r}} \Psi_{x_i}\left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \dots, rv_n\right) \tag{3.11}$$

We now observe that for $r \leqslant r_1$ and $(v_2, v_3, \dots, v_n)$ satisfying (3.7)

$$\left(\sum_{k=2}^{n} T_k^2\right)^{1/2} \leqslant \frac{Kr}{|\mu_2^1|}. \tag{3.12}$$

Equation (3.12) implies that for

$$r \leqslant \min\left(r_1, \frac{|\mu_2^1| \epsilon_1}{K}\right) \equiv r_2 \tag{3.13}$$

$$T(\cdot, r) : \sum_{k=2}^{n} v_k^2 \leqslant \epsilon_2^2 \to \sum_{k=2}^{n} v_k^2 \leqslant \epsilon_1^2; \tag{3.14}$$

hence for $r \leqslant r_2$ any fixed point of (3.10) in $\sum_{k=2}^{n} v_k^2 \leqslant \epsilon_2^2$ must be in $\sum_{k=2}^{n} v_k^2 \leqslant \epsilon_1^2$. The smoothness of Ψ implies that $T(\cdot, r)$ is C^1 in $(v_2, v_3, \dots, v_n)$ and hence Brouwer's Theorem guarantees (for all $r \leqslant r_2$) the existence of at least one solution of (3.10) (and hence (3.3)) in $\sum_{k=2}^{n} v_k^2 \leqslant \epsilon_2^2$.

To establish uniqueness, it suffices to show that for some $r_0 \leqslant r_2$ and all r in $[0, r_0]$ the maps $T(\cdot, r)$ are contractions on $\sum_{k=2}^{n} v_k^2 \leqslant \epsilon_2^2$. This computation follows from (2.1).

The smoothness of

$$r \to \mathbf{v}_1(r) \equiv \left(\sqrt{1 - \sum_{k=2}^{n} v_k^2(r)}, v_2(r), \dots, v_n(r)\right)$$

follows from the smoothness of $\mathbf{x} \to \Psi(\mathbf{x})$. We find that for $0 < r \leqslant r_0$ $\dot{v}_i(r) \equiv (d/dr)\, v_i(r)$ exists, is continuous, and satisfies

$$\sum_{j=2}^{n} B_{ij}(v_2, v_3, \dots, v_n, r)\, \dot{v}_j(r) = F_i(v_2, v_3, \dots, v_n; r), \qquad i = 2, 3, \dots, n, \tag{3.15}$$

where

$$B_{ij}(v_2, v_3, \ldots, v_n; r)$$

$$= \left\{ 2\mu_i^1 - \frac{\Psi_{x_1}}{r\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} - \frac{\{\Psi_{x_1x_j}v_i + \Psi_{x_1x_i}v_j\}}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}}$$

$$+ \frac{v_iv_j}{\left(1 - \sum_{k=2}^{n} v_k^2\right)^{3/2}} \left\{ \sqrt{1 - \sum_{k=2}^{n} v_k^2}\, \Psi_{x_1x_1} - \frac{\Psi_{x_1}}{r} \right\} + \Psi_{x_ix_j};$$

$$i, j, = 2, 3, \ldots, n, \tag{3.16}$$

and

$$F_i(v_2, v_3, \ldots, v_n; r)$$

$$\equiv \frac{1}{r^2} \left\{ \Psi_{x_i} - \frac{\Psi_{x_1}v_i}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} + \frac{v_i}{r} \left\{ \Psi_{x_1x_1} + \frac{\sum_{j=2}^{n} \Psi_{x_1x_j}v_j}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\}$$

$$+ \frac{1}{r} \left\{ \sqrt{1 - \sum_{k=2}^{n} v_k^2}\, \Psi_{x_1x_i} + \sum_{j=2}^{n} \Psi_{x_ix_j}v_j \right\}, \qquad i = 2, 3, \ldots, n. \tag{3.17}$$

To obtain the limiting relation (3.4) we simply make use of the estimate (3.12).

To establish Theorem 2 it again suffices to work with a particular branch of critical points of ϕ. We shall extend the branch $\mathbf{x}_1^+(r) \equiv r\mathbf{v}_1^+(r)$, $r \in [0, r_0]$. It is clear that if we extend $\mathbf{v}_1^+(r)$ as a solution the initial value problem (2.7), then $\mathbf{x}_1^+(r) \equiv r\mathbf{v}_1^+(r)$ will be a critical point of ϕ on $\mathbf{x} \cdot \mathbf{x} = r^2$. It is also clear that the initial value problem (or any of its representations) has a (have) unique solution(s) provided the operator $\mathscr{B}(r\mathbf{v}_1^+(r)) : V(\mathbf{v}_1^+(r)) \to V(\mathbf{v}_1^+(r))$ is invertible. The condition for the lack of invertibility $\mathscr{B}$ along $r\mathbf{v}_1^+(r)$ is simply that the quadratic $B(r\mathbf{v}_1^+(r); \mathbf{u}, \mathbf{u})$ have 0 as a critical value on the unit sphere $V(\mathbf{v}_1^+(r))$. That $B(r\mathbf{v}_1^+(r); \mathbf{u}, \mathbf{u})$ has no positive and $n - 1$ negative critical values for $r < R_1$ follows from the fact that $\mathscr{B}(r\mathbf{v}_1^+(r))$ is symmetric and invertible for $r < R_1$ and the fact that

$$\mathscr{B}_{0,1} \equiv \lim_{r \to 0^+} \mathscr{B}(r\mathbf{v}_1^+(r)) = \operatorname{diag}(\mu_2^1, \mu_3^1, \ldots, \mu_n^1)$$

maps $V(\mathbf{e}_1) \to V(\mathbf{e}_1)$ and has no positive and $n - 1$ negative eigenvalues.